On certain generalizations of twisted affine Lie algebras and quasimodules for Γ-vertex algebras

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Abstract

We continue a previous study on Γ-vertex algebras and their quasimodules. In this paper we refine certain known results and we prove that for any $\mathbb{Z}$-graded vertex algebra V and a positive integer N, the category of V-modules is naturally isomorphic to the category of quasimodules of a certain type for $V^{\otimes N}$. We also study certain generalizations of twisted affine Lie algebras and we relate such Lie algebras to vertex algebras and their quasimodules, in a way similar to that twisted affine Lie algebras are related to vertex algebras and their twisted modules.

1 Introduction

It has been fairly well known that infinite-dimensional Lie algebras such as (untwisted) affine Kac-Moody Lie algebras and the Virasoro Lie algebra through their highest weight modules can be associated with vertex operator algebras and modules (cf. [FZ], [DL]). On the other hand, it was known (see [FLM], [Li2]) that twisted affine Lie algebras through their highest weight modules can also be associated with vertex operator algebras and their twisted modules.

In [GKK], Golenishcheva-Kutuzova and Kac introduced and studied a notion of Γ-conformal algebra with Γ a group. As it was proved therein, a Γ-conformal algebra structure on a vector space $\mathfrak{g}$ exactly amounts to a Lie algebra structure on $\mathfrak{g}$ together with a group action of Γ on $\mathfrak{g}$ by automorphisms such that for any $a, b \in \mathfrak{g}$, $[ga, b] = 0$ for all but finitely many $g \in \Gamma$. To each Γ-conformal algebra $\mathfrak{g}$, they associated an infinite-dimensional Lie algebra whose underlying vector space is a certain quotient space of $\mathfrak{g} \otimes \mathbb{C}[t, t^{-1}]$. Several families of infinite-dimensional Lie algebras, including centerless twisted affine Lie algebras and quantum torus Lie algebras (see [GKL1-2]), were realized in terms of Γ-conformal algebras, and some new infinite-dimensional Lie algebras were also constructed.

In [Li3], to associate vertex algebra-like structures to Lie algebras like quantum torus Lie algebras, we studied "quasilocal" subsets of $\mathrm{Hom}(W, W((x)))$ for any given vector space W and we proved that any quasilocal subset generates a vertex algebra in a certain canonical way. However, the vector space W under the obvious action is *not* a module for

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the vertex algebras generated by quasilocal subsets. Then a new notion of what we called a quasimodule naturally arose. For a vertex algebra V, a quasimodule is a vector space W equipped with a linear map Y_W from V to $\mathrm{Hom}(W, W((x)))$ satisfying the condition that $Y_W(\mathbf{1}, x) = 1$ and for $u, v \in V$, there exists a nonzero polynomial $p(x_1, x_2)$ such that

$$\begin{aligned}
& x_0^{-1}\delta\left(\frac{x_1 - x_2}{x_0}\right) p(x_1, x_2) Y_W(u, x_1) Y_W(v, x_2) \\
& \qquad - x_0^{-1}\delta\left(\frac{x_2 - x_1}{-x_0}\right) p(x_1, x_2) Y_W(v, x_2) Y_W(u, x_1) \\
= \; & x_2^{-1}\delta\left(\frac{x_1 - x_0}{x_2}\right) p(x_1, x_2) Y_W(Y(u, x_0)v, x_2).
\end{aligned}$$

In terms of this notion, any vector space W is a quasimodule for the vertex algebras generated by quasilocal subsets. Taking W to be a highest weight module for the quantum torus Lie algebra we obtain a vertex algebra with W as a quasimodule.

For a vertex algebra V, the notion of quasimodule is intrinsically related to the notion of twisted module (with respect to a finite order automorphism). As we mentioned before, highest weight modules for twisted affine Lie algebras are naturally twisted modules for the vertex operator algebras associated to untwisted affine Lie algebras. It was showed in [Li3] that highest weight modules of fixed level for a twisted affine Lie algebra, which is viewed as an invariant subalgebra of the untwisted affine Lie algebra, are naturally quasimodules for the vertex algebras associated to an untwisted affine Lie algebra. Motivated by these two facts, in [Li4], by using a result of Barron, Dong and Mason [BDM], we established a canonical connection between twisted modules and quasimodules for general vertex operator algebras.

Let V be a vertex operator algebra and let σ be an order-N automorphism of V. For a σ-twisted V-module W, the vertex operator map Y_W, a linear map from V to $\mathrm{Hom}(W, W((x^{1/N})))$, satisfies the following invariance property

$$Y_W(\sigma v, x) = \lim_{x^{1/N} \to \omega_N x^{1/N}} Y_W(v, x) \quad \text{for } v \in V.$$

Set $\Gamma = \langle \sigma \rangle \subset \mathrm{Aut} V$ and let $\phi : \Gamma \to \mathbb{C}^\times$ is the group embedding determined by $\phi(\sigma) = \exp(2\pi i/N)$. We call a quasimodule (W, Y_W) for V a (Γ, ϕ)-*quasimodule* if for any $u, v \in V$, there exists a nonnegative integer k such that the quasi-Jacobi identity holds with $p(x_1, x_2) = (x_1^N - x_2^N)^k$ and such that the following invariance property holds:

$$Y_W(gv, x) = Y_W(\phi(g)^{L(0)} v, \phi(g)x) \quad \text{for } v \in V, \; g \in \Gamma.$$

What was proved in [Li4] is that the category of (weak) σ-twisted V-modules is naturally isomorphic to the category of (Γ, ϕ)-quasimodules for V.

In [Li3], partially motivated by the notion of Γ-conformal algebra in [GKK] we formulated and studied a notion of Γ-vertex algebra. For any group Γ, a Γ-vertex algebra can be equivalently defined as a vertex algebra V equipped with two group homomorphisms

$$R : \Gamma \to GL(V), \quad \phi : \Gamma \to \mathbb{C}^\times$$

such that $R_g\mathbf{1} = \mathbf{1}$ for $g \in \Gamma$ and

$$R_g Y(v,x) R_g^{-1} = Y(R_g v, \phi(g)^{-1}x) \quad \text{for } g \in \Gamma,\ v \in V.$$

Let V be a $\mathbb{Z}$-graded vertex algebra with $L(0)$ the grading operator and let Γ be a group of automorphisms of $\mathbb{Z}$-graded vertex algebra V. Let $\phi : \Gamma \to \mathbb{C}^\times$ be any group homomorphism. Define $R_g = \phi(g)^{-L(0)}g$ for $g \in \Gamma$. Then V becomes a Γ-vertex algebra.

In this paper, we formulate and study a notion of quasimodule for Γ-vertex algebras. For a Γ-vertex algebra V, a V-quasimodule W is a quasimodule for V viewed as a vertex algebra such that for any $u, v \in V$, the quasi-Jacobi identity holds with $p(x_1, x_2) = (x_1 - \alpha_1 x_2) \cdots (x_1 - \alpha_r x_2)$ for some $\alpha_1, \ldots, \alpha_r \in \phi(\Gamma) \subset \mathbb{C}^\times$ and such that

$$Y_W(R_g v, x) = Y_W(v, \phi(g)x) \quad \text{for } v \in V,\ g \in \Gamma.$$

Note that for any vector space W, the group $\mathbb{C}^\times$ acts on the space $\mathrm{Hom}(W, W((x)))$ by

$$R_\lambda a(x) = a(\lambda x) \quad \text{for } a(x) \in \mathrm{Hom}(W, W((x))),\ \lambda \in \mathbb{C}^\times.$$

Let Γ be a subgroup of $\mathbb{C}^\times$. A subset S of $\mathrm{Hom}(W, W((x)))$ is said to be Γ-local (see [GKK]) if for any $a(x), b(x) \in S$, there there exists $\alpha_1, \ldots, \alpha_r \in \Gamma$ such that

$$(x_1 - \alpha_1 x_2) \cdots (x_1 - \alpha_r x_2)[a(x_1), b(x_2)] = 0.$$

By refining a result of [Li3], we prove that every Γ-local subset of $\mathrm{Hom}(W, W((x)))$ generates a Γ-vertex algebra with W as a quasimodule. We also obtain an analogue of Borcherds' commutator formula for quasimodules.

A conceptual result of Barron, Dong and Mason [BDM] is that for any vertex operator algebra V and for any positive integer N, the category of (weak) V-modules is canonically isomorphic to the category of (weak) σ-twisted $V^{\otimes N}$-modules, where σ is a permutation automorphism of $V^{\otimes N}$. In this paper we show that for any $\mathbb{Z}$-graded vertex algebra V and for any positive integer N, a V-module structure on a vector space W exactly amounts to a quasimodule structure for $V^{\otimes N}$ viewed as a Γ-vertex algebra with $\Gamma = \langle (12 \cdots N) \rangle$, where $(12 \cdots N)$ denotes the permutation automorphism of $V^{\otimes N}$. This result can be considered as a version of Barron, Dong and Mason's theorem in terms of quasimodules.

As we mentioned earlier, Golenishcheva-Kutuzova and Kac gave a construction of infinite-dimensional Lie algebras from Γ-conformal algebras (in the sense of [GKK]). In this paper, we slightly extend their construction with a central extension being included. Let $\mathfrak{g}$ be any Lie algebra equipped with a symmetric invariant bilinear form $\langle \cdot, \cdot \rangle$. Associated to the pair $(\mathfrak{g}, \langle \cdot, \cdot \rangle)$, one has the untwisted affine Lie algebra $\hat{\mathfrak{g}} = \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}\mathbf{k}$ (see [K1]). Let Γ be a group of automorphisms of $\mathfrak{g}$, preserving the bilinear form, such that for any $a, b \in \mathfrak{g}$,

$$[ga, b] = 0, \qquad \langle ga, b \rangle = 0 \quad \text{for all but finitely many } g \in \Gamma,$$

let $\phi : \Gamma \to \mathbb{C}^\times$ be any group homomorphism. We construct a Lie algebra $\hat{\mathfrak{g}}[\Gamma]$ as a quotient space of $\hat{\mathfrak{g}}$. Furthermore, we prove that the category of "restricted" $\hat{\mathfrak{g}}[\Gamma]$-modules of level ℓ

is canonically isomorphic to the category of quasimodules for $V_{\hat{\bar{\mathfrak{g}}}}(\ell,0)$ viewed as a Γ-vertex algebra, where $\bar{\mathfrak{g}}$ is a certain Lie algebra with $\bar{\mathfrak{g}}=\mathfrak{g}$ for ϕ an injective homomorphism. We also extend Golenishcheva-Kutuzova and Kac's notion of Γ-conformal algebra to include higher order singularity.

Note that extended affine Lie algebras (cf. [S1,2], [MY], [AABGP]) form a relatively large family of Lie algebras, including finite-dimensional simple Lie algebras, (twisted and twisted) affine Lie algebras, toroidal Lie algebras, and quantum torus algebras. As these special extended affine Lie algebras have been associated with vertex algebras ([FZ], [BBS], [Li3]), our naive hope is that every extended affine Lie algebra can be realized as a generalized twisted affine Lie algebra $\hat{\mathfrak{g}}[\Gamma]$, so that all the extended affine Lie algebras can be associated with vertex algebras and quasimodules.

This paper is organized as follows: In Section 2, we reformulate and refine certain results on Γ-vertex algebras and quasimodules. In Section 3, we give a natural isomorphism between the category of V-modules and a certain subcategory of quasimodules for $V^{\otimes N}$. In Section 4, we study certain generalizations of twisted affine Lie algebras and we relate them with vertex algebras and quasimodules.

2 Γ-vertex algebras and their quasimodules

In this section we reformulate the notion of Γ-vertex algebra of [Li3] and we define a notion of quasimodule for a Γ-vertex algebra. Certain results of [Li3] are refined and an analogue of Borcherds' commutator formula is obtained.

We first recall from [Li3] the notion of quasimodule for a vertex algebra V, which generalizes the notion of module. A V*-quasimodule* is a vector space W equipped with a linear map

$$Y_W : V \to \mathrm{Hom}(W, W((x))) \subset (\mathrm{End}\, W)[[x, x^{-1}]]$$

satisfying the condition that $Y_W(\mathbf{1}, x) = 1$ and for $u, v \in V$, there exists a nonzero polynomial $p(x_1, x_2)$ such that

$$\begin{aligned}
&x_0^{-1}\delta\left(\frac{x_1-x_2}{x_0}\right)p(x_1,x_2)Y_W(u,x_1)Y_W(v,x_2)\\
&\qquad\qquad -x_0^{-1}\delta\left(\frac{x_2-x_1}{-x_0}\right)p(x_1,x_2)Y_W(v,x_2)Y_W(u,x_1)\\
&= x_2^{-1}\delta\left(\frac{x_1-x_0}{x_2}\right)p(x_1,x_2)Y_W(Y(u,x_0)v,x_2). \qquad (2.1)
\end{aligned}$$

Lemma 2.1. *Let V be a vertex algebra. A V-quasimodule (W, Y_W) is a V-module if and only if for $u, v \in V$, $Y_W(u, x)$ and $Y_W(v, x)$ are mutually local in the sense that*

$$(x_1-x_2)^k Y_W(u,x_1)Y_W(v,x_2) = (x_1-x_2)^k Y_W(v,x_2)Y_W(u,x_1)$$

for some nonnegative integer k, depending on u and v.

Proof. We only need to prove the "if" part. For $u, v \in V$, there exists a nonzero polynomial $p(x_1, x_2)$ such that (2.1) holds. Using the delta-function substitution we have

$$\begin{aligned}
& p(x_0+x_2, x_2) \\
& \cdot \left(x_0^{-1}\delta\left(\frac{x_1-x_2}{x_0}\right) Y_W(u,x_1)Y_W(v,x_2) - x_0^{-1}\delta\left(\frac{x_2-x_1}{-x_0}\right) Y_W(v,x_2)Y_W(u,x_1) \right) \\
= \;& p(x_0+x_2, x_2) x_2^{-1}\delta\left(\frac{x_1-x_0}{x_2}\right) Y_W(Y(u,x_0)v, x_2). \qquad (2.2)
\end{aligned}$$

As $Y_W(u,x)$ and $Y_W(v,x)$ are mutually local, the second factor of the left-hand side of (2.2) involves only finitely many negative powers of x_0. Recall from [Li3] the natural field-embedding

$$\iota_{x_1,x_2} : \mathbb{C}(x_1,x_2) \to \mathbb{C}((x_1))((x_2)),$$

where $\mathbb{C}(x_1, x_2)$ denotes the field of rational functions. Then we multiply both sides of (2.2) by $\iota_{x_2,x_0}(1/p(x_0 + x_2, x_2))$, obtaining the usual Jacobi identity. Thus (W, Y_W) is a V-module. $\square$

Let Γ be a group which is fixed throughout this section, and denote by $\mathbb{C}^\times$ the group of nonzero complex numbers.

Definition 2.2. A Γ-*vertex algebra* is a vertex algebra V equipped with group homomorphisms

$$R : \Gamma \to \mathrm{GL}(V);\; g \mapsto R_g \qquad (2.3)$$

$$\phi : \Gamma \to \mathbb{C}^\times \qquad (2.4)$$

such that $R_g(\mathbf{1}) = \mathbf{1}$ for $g \in \Gamma$ and

$$R_g Y(v,x) R_g^{-1} = Y(R_g(v), \phi(g)^{-1}x) \quad \text{for } g \in \Gamma,\; v \in V. \qquad (2.5)$$

In view of ([Li3], Theorem 6.5), this notion is equivalent to the notion of Γ-vertex algebra defined in [Li3].

Example 2.3. Let V be a $\mathbb{Z}$-graded vertex algebra in the sense that V is a vertex algebra equipped with a $\mathbb{Z}$-grading $V = \coprod_{n\in\mathbb{Z}} V_{(n)}$ such that

$$u_k V_{(n)} \subset V_{(m+n-k-1)} \quad \text{for } u \in V_{(m)},\; m,n,k \in \mathbb{Z}.$$

Denote by $L(0)$ the grading operator, i.e.,

$$L(0)v = nv \quad \text{for } v \in V_{(n)},\; n \in \mathbb{Z}.$$

Let Γ be a group of grading-preserving automorphisms of V and let ϕ be any group homomorphism from Γ to $\mathbb{C}^\times$. Define $R : \Gamma \to \mathrm{GL}(V)$ by

$$R_g(v) = \phi(g)^{-L(0)}(gv) \quad \text{for } g \in \Gamma,\; v \in V. \qquad (2.6)$$

From [Li3], V becomes a Γ-vertex algebra.

Remark 2.4. Let V be a Γ-vertex algebra and let $g \in \ker R$. We have $Y(v,x)\mathbf{1} = Y(v,\phi(g)^{-1}x)\mathbf{1}$ for $v \in V$. Thus $e^{x\mathcal{D}}v = e^{\phi(g)^{-1}x\mathcal{D}}v$. If $\mathcal{D} \neq 0$, or equivalently, if V is not a classical commutative associative algebra, we have $\phi(g) = 1$. That is, if $\mathcal{D} \neq 0$, we have $\ker R \subset \ker \phi$, so we can replace Γ by the quotient group $\Gamma/\ker R$ with a faithful action on V.

Definition 2.5. Let V be a Γ-vertex algebra. A *V-quasimodule* is a quasimodule (W, Y_W) for V viewed as a vertex algebra, satisfying the condition that

$$Y_W(R_g v, x) = Y_W(v, \phi(g)x) \quad \text{for } g \in \Gamma,\ v \in V, \tag{2.7}$$

and for $u, v \in V$, there exist $\alpha_1, \dots, \alpha_k \in \phi(\Gamma) \subset \mathbb{C}^\times$ such that

$$(x_1 - \alpha_1 x_2) \cdots (x_1 - \alpha_k x_2)[Y_W(u,x_1), Y_W(v,x_2)] = 0. \tag{2.8}$$

Remark 2.6. Let V be a vertex operator algebra in the sense of [FLM] and let σ be an order-N automorphism of V. Set $\Gamma = \langle \sigma \rangle$ and let ϕ be the group homomorphism from Γ to $\mathbb{C}^\times$, defined by $\phi(\sigma) = \exp(2\pi i/N)$. Consider V as a Γ-vertex algebra as in Example 2.3. It was proved in [Li4] that the category of weak σ-twisted V-modules is isomorphic to the category of quasimodules for V viewed as a Γ-vertex algebra.

Now, let W be a general vector space. Set

$$\mathcal{E}(W) = \mathrm{Hom}(W, W((x))) \subset (\mathrm{End}\, W)[[x, x^{-1}]].$$

A subset S of $\mathcal{E}(W)$ is said to be *quasi-local* if for any $a(x), b(x) \in S$, there exists a nonzero polynomial $p(x_1, x_2)$ such that

$$p(x_1,x_2)a(x_1)b(x_2) = p(x_1,x_2)b(x_2)a(x_1). \tag{2.9}$$

Let Γ be a subgroup of $\mathbb{C}^\times$. Following [GKK], we say $a(x), b(x) \in \mathcal{E}(W)$ are Γ-*local* if

$$(x_1 - \alpha_1 x_2) \cdots (x_1 - \alpha_r x_2)a(x_1)b(x_2) = (x_1 - \alpha_1 x_2) \cdots (x_1 - \alpha_r x_2)b(x_2)a(x_1) \tag{2.10}$$

for some $\alpha_1, \dots, \alpha_r \in \Gamma$. A subset S of $\mathcal{E}(W)$ is said to be Γ-*local* if every pair in S is Γ-local.

Remark 2.7. Note that $\mathcal{E}(W)$ is naturally a vector space over the field $\mathbb{C}((x))$. On the other hand, we define a group action of $\mathbb{C}^\times$ on $\mathcal{E}(W)$ with $\alpha \in \mathbb{C}^\times$ acting as R_α by

$$R_\alpha a(x) = a(\alpha x) \quad \text{for } a(x) \in \mathcal{E}(W) \tag{2.11}$$

(cf. [GKK]). Notice that for any quasi-local subset S of $\mathcal{E}(W)$, the subspace spanned by $R_\alpha(S)$ for $\alpha \in \mathbb{C}^\times$ is also quasi-local. Consequently, every maximal quasi-local subspace of $\mathcal{E}(W)$ is closed under the action of $\mathbb{C}^\times$.

Assume that $a(x), b(x) \in \mathcal{E}(W)$ are quasi-local. Notice that the commutativity relation (2.9) implies

$$p(x_1, x_2)a(x_1)b(x_2) \in \mathrm{Hom}(W, W((x_1, x_2))).$$

We define

$$\begin{aligned} Y_{\mathcal{E}}(a(x), x_0)b(x) &= \iota_{x,x_0}(1/p(x + x_0, x))\mathrm{Res}_{x_1}x_1^{-1}\delta\left(\frac{x + x_0}{x_1}\right)(p(x_1, x)a(x_1)b(x)) \\ &= \iota_{x,x_0}(1/p(x + x_0, x))\left(p(x_1, x)a(x_1)b(x)\right)|_{x_1=x+x_0}. \end{aligned}$$

Write

$$Y_{\mathcal{E}}(a(x), x_0)b(x) = \sum_{n\in\mathbb{Z}} a(x)_n b(x) x_0^{-n-1}.$$

A quasilocal subspace U of $\mathcal{E}(W)$ is said to be *closed* (under $Y_{\mathcal{E}}$) if

$$a(x)_n b(x) \quad \text{ for } a(x), b(x) \in U,\ n \in \mathbb{Z}. \tag{2.12}$$

Remark 2.8. For every $\alpha \in \mathbb{C}^\times$, $Y_{\mathcal{E}}^{(\alpha)}(a(x), x_0)b(x)$ was defined in [Li3], where

$$Y_{\mathcal{E}}^{(\alpha)}(a(x), x_0)b(x) = \iota_{x,x_0}(1/p(\alpha x + x_0, x))\left(p(x_1, x)a(x_1)b(x)\right)|_{x_1=\alpha x+x_0}.$$

It was proved ([Li3], Proposition 3.9) that

$$Y_{\mathcal{E}}^{(\alpha)}(a(x), x_0)b(x) = Y_{\mathcal{E}}(a(\alpha x), \alpha^{-1}x_0)b(x).$$

It is clear that a quasilocal subspace U of $\mathcal{E}(W)$ is closed under all the operations $Y_{\mathcal{E}}^{(\alpha)}$ for $\alpha \in \Gamma$ (a subgroup of $\mathbb{C}^\times$) if and only if U is closed under $Y_{\mathcal{E}}$ and closed under the action of Γ.

Theorem 2.9. *Let W be a vector space over $\mathbb{C}$ and let Γ be a subgroup of $\mathbb{C}^\times$. For any Γ-local subset S of $\mathcal{E}(W)$, there exist closed Γ-local subspaces K of $\mathcal{E}(W)$ with the following property*

$$\{1_W\} \cup S \subset K, \quad R_\alpha a(x) = a(\alpha x) \in K \quad \textit{for } \alpha \in \Gamma,\ a(x) \in K, \tag{2.13}$$

among which the smallest subspace is denoted by $\langle S\rangle_\Gamma$. Furthermore, let V be any closed Γ-local subspace of $\mathcal{E}(W)$ such that

$$1_W \in V \ \textit{ and } \ R_\alpha a(x) = a(\alpha x) \in V \quad \textit{for } \alpha \in \Gamma,\ a(x) \in V.$$

Then $(V, 1_W, Y_{\mathcal{E}})$ carries the structure of a Γ-vertex algebra with W as a quasimodule where $Y_W(a(x), x_0) = a(x_0)$ for $a(x) \in V$.

Proof. By ([Li3], Corollary 4.7), there exists a (unique) smallest Γ-local subspace $\langle S\rangle_\Gamma$, that contains 1_W and S and that is closed under $Y_{\mathcal{E}}^{(\alpha)}$ for $\alpha \in \Gamma$. From Remark 2.8, $\langle S\rangle_\Gamma$ is also the smallest Γ-local subspace that is closed under $Y_{\mathcal{E}}$ and under the action of Γ. This proves the first assertion.

For the second assertion, with V being Γ-local, V is quasilocal. By Theorems 6.3 and 6.5 of [Li3], $(V, 1_W, Y_{\mathcal{E}})$ carries the structure of a Γ-vertex algebra with W as a quasimodule where $Y_W(\alpha(x), x_0) = \alpha(x_0)$. For $\alpha \in \Gamma,\ a(x) \in V$, we have

$$Y_W(R_\alpha a(x), x_0) = Y_W(a(\alpha x), x_0) = a(\alpha x_0) = Y_W(a(x), \alpha x_0).$$

Thus, (W, Y_W) is a V-quasimodule. $\square$

Remark 2.10. Notice that for $\Gamma = \{1\}$ (the trivial group), Γ-locality becomes the usual locality (cf. [Li1]). Assume that $a(x), b(x) \in \mathcal{E}(W)$ are local, i.e., there exists a nonnegative integer k such that

$$(x_1 - x_2)^k a(x_1) b(x_2) = (x_1 - x_2)^k b(x_2) a(x_1).$$

Then

$$\begin{aligned}
& Y_{\mathcal{E}}(a(x), x_0) b(x) \\
= \;& x_0^{-k} \mathrm{Res}_{x_1} x_1^{-1} \delta\left(\frac{x + x_0}{x_1}\right) \left((x_1 - x)^k a(x_1) b(x)\right) \\
= \;& \mathrm{Res}_{x_1} x_0^{-1} \delta\left(\frac{x_1 - x}{x_0}\right) x_0^{-k} \left((x_1 - x)^k a(x_1) b(x)\right) \\
& - \mathrm{Res}_{x_1} x_0^{-1} \delta\left(\frac{x - x_1}{-x_0}\right) x_0^{-k} \left((x_1 - x)^k a(x_1) b(x)\right) \\
= \;& \mathrm{Res}_{x_1} x_0^{-1} \delta\left(\frac{x_1 - x}{x_0}\right) (x_1 - x)^{-k} \left((x_1 - x)^k a(x_1) b(x)\right) \\
& - \mathrm{Res}_{x_1} x_0^{-1} \delta\left(\frac{x - x_1}{-x_0}\right) (-x + x_1)^{-k} \left((x_1 - x)^k b(x) a(x_1)\right) \\
= \;& \mathrm{Res}_{x_1} \left(x_0^{-1} \delta\left(\frac{x_1 - x}{x_0}\right) a(x_1) b(x) - x_0^{-1} \delta\left(\frac{x - x_1}{-x_0}\right) b(x) a(x_1) \right),
\end{aligned}$$

which agrees with the definition given in [Li1]. Also, for $\Gamma = \{1\}$, a quasimodule is simply a module. Thus Theorem 2.9 generalizes the corresponding result of [Li1].

Proposition 2.11. *Let V be a Γ-vertex algebra and let W be a vector space equipped with a linear map Y_W from V to $\mathcal{E}(W)$ such that $Y_W(\mathbf{1}, x) = 1_W$. Set*

$$\overline{V} = \{Y_W(v, x) \mid v \in V\} \subset \mathcal{E}(W).$$

Then (W, Y_W) carries the structure of a V-quasimodule if and only if $\overline{V}$ is $\phi(\Gamma)$-local, closed, and the map Y_W is a homomorphism of vertex algebras from V to $(\overline{V}, 1_W, Y_{\mathcal{E}})$ such that

$$Y_W(R_g v, x) = Y_W(v, \phi(g)x) \quad \textit{for } g \in \Gamma.$$

Proof. It was proved ([Li3], Proposition 5.4) that (W, Y_W) carries the structure of a quasimodule for V viewed as a vertex algebra if and only if $\overline{V}$ is quasi-local and closed, and Y_W is a vertex-algebra homomorphism from V to $(\overline{V}, 1_W, Y_{\mathcal{E}})$. Then it follows immediately. $\square$

Proposition 2.12. *Let V be a Γ-vertex algebra. Denote by J the ideal of V generated by the elements $R_h v - v$ for $h \in \ker \phi \subset \Gamma,\ v \in V$ and set $\overline{V} = V/J$ (the quotient vertex algebra). Then $\ker \phi$ acts trivially on $\overline{V}$ and $\overline{V}$ is a $\Gamma/\ker \phi$-vertex algebra with injective group homomorphism from $\Gamma/\ker\phi$ to $\mathbb{C}^{\times}$. Furthermore, for any V-quasimodule (W, Y_W), $Y_W(u,x) = 0$ for $u \in J$ and W is naturally a $\overline{V}$-quasimodule.*

Proof. For $g \in \Gamma,\ h \in \ker\phi,\ v \in V$, we have

$$R_g(R_h v - v) = R_{ghg^{-1}} R_g v - R_g v \in J,$$

as $ghg^{-1} \in \ker\phi$. From [LL], J is linearly spanned by all the coefficients of the formal series

$$Y(u,x)(R_h v - v), \quad Y(R_h v - v, x)u$$

for $h \in \ker\phi,\ u, v \in V$. As $R_g Y(a,x)b = Y(R_g a, \phi(g)x)R_g b$ for $g \in \Gamma,\ a, b \in V$, it follows that J is stable under the action of Γ. From definition, $\ker\phi$ acts trivially on $\overline{V}$. Then the first assertion is clear. Now let (W, Y_W) be a V-quasimodule. For $h \in \ker\phi,\ v \in V$, we have

$$Y_W(R_h v, x) = Y_W(v, \phi(h)x) = Y_W(v, x).$$

Thus $R_h v - v \in \ker Y_W$ for $h \in \ker\phi,\ v \in V$. As Y_W is a homomorphism of vertex algebras (Proposition 2.11), $\ker Y_W$ is an ideal of V. Consequently, $J \subset \ker Y_W$. Then W is naturally a $\overline{V}$-quasimodule. $\square$

For quasimodules we have the following analogue of Borcherds' commutator formula:

Proposition 2.13. *Let V be a Γ-vertex algebra and let (W, Y_W) be a V-quasimodule. Let $\psi : \phi(\Gamma) \to \Gamma$ be any section of ϕ. For $u, v \in V$, we have*

$$[Y_W(u,x_1), Y_W(v,x_2)] = \sum_{\alpha \in \phi(\Gamma)} \mathrm{Res}_{x_0} x_1^{-1} \delta\left(\frac{\alpha x_2 + x_0}{x_1}\right) Y_W(Y(R_{\psi(\alpha)}u, \alpha^{-1}x_0)v, x_2), \tag{2.14}$$

which is a finite sum.

Proof. Set $\overline{V} = \{Y_W(v,x) \mid v \in V\} \subset \mathcal{E}(W)$. By Proposition 2.11, $\overline{V}$ is $\phi(\Gamma)$-local and closed and Y_W is a vertex-algebra homomorphism from V to $(\overline{V}, 1_W, Y_{\mathcal{E}})$. Recall from [Li3] (Corollary 3.12) that if $a(x), b(x) \in \mathcal{E}(W)$ satisfy the following relation

$$(x_1 - \alpha_1 x_2)^{k_1} \cdots (x_1 - \alpha_r x_2)^{k_r} [a(x_1), b(x_2)] = 0,$$

where $\alpha_1, \ldots, \alpha_r$ are distinct nonzero complex numbers and $k_1, \ldots, k_r$ are nonnegative integers, then

$$Y_{\mathcal{E}}(a(\beta x), x_0)b(x) \in \mathcal{E}(W)[[x_0]] \quad \text{for } \beta \notin \{\alpha_1, \ldots, \alpha_r\}$$

and

$$[a(x_1), b(x_2)] = \sum_{\beta \in \mathbb{C}^{\times}} \mathrm{Res}_{x_0} x_1^{-1} \delta\left(\frac{\beta x_2 + x_0}{x_1}\right) Y_{\mathcal{E}}(a(\beta x_2), \beta^{-1}x_0)b(x_2). \tag{2.15}$$

(Note that $Y_{\mathcal{E}}^{(\beta)}(a(x), x_0)b(x) = Y_{\mathcal{E}}(a(\beta x), \beta^{-1}x_0)b(x)$.) For $u, v \in V$, using the fact that Y_W is a vertex-algebra homomorphism, we get

$$
\begin{aligned}
& [Y_W(u, x_1), Y_W(v, x_2)] \\
= & \sum_{\beta \in \phi(\Gamma)} \mathrm{Res}_{x_0} x_1^{-1} \delta\left(\frac{\beta x_2 + x_0}{x_1}\right) \left(Y_{\mathcal{E}}(Y_W(u, \beta x), \beta^{-1}x_0) Y_W(v, x)\right)|_{x=x_2} \\
= & \sum_{\beta \in \phi(\Gamma)} \mathrm{Res}_{x_0} x_1^{-1} \delta\left(\frac{\beta x_2 + x_0}{x_1}\right) \left(Y_{\mathcal{E}}(Y_W(R_{\psi(\beta)}u, x), \beta^{-1}x_0) Y_W(v, x)\right)|_{x=x_2} \\
= & \sum_{\beta \in \phi(\Gamma)} \mathrm{Res}_{x_0} x_1^{-1} \delta\left(\frac{\beta x_2 + x_0}{x_1}\right) Y_W\left(Y(R_{\psi(\beta)}u, \beta^{-1}x_0)v, x_2\right),
\end{aligned}
$$

which is a finite sum. ∎

Corollary 2.14. *Let V be a Γ-vertex algebra with a faithful action of Γ. Suppose that V has a faithful quasimodule. Then Γ must be abelian and $\phi : \Gamma \to \mathbb{C}^{\times}$ is injective. Furthermore, for any $u, v \in V$,*

$$Y(R_g u, x)v \in V[[x]] \quad \text{for all but finitely many } g \in \Gamma, \tag{2.16}$$

or equivalently,

$$[Y(R_g u, x_1), Y(v, x_2)] = 0 \quad \text{for all but finitely many } g \in \Gamma. \tag{2.17}$$

Proof. Let (W, Y_W) be a faithful V-quasimodule. Let $g \in \ker \phi$. Then

$$Y_W(R_g v, x) = Y_W(v, x) \quad \text{for all } v \in V.$$

It follows that $R_g = 1$. As Γ acts faithfully on V, we have $g = 1$. This proves that ϕ is injective. Consequently, Γ is abelian.

For $u, v \in V,\ g \in \Gamma$, we have

$$Y_W(Y(R_g u, x_0)v, x) = Y_{\mathcal{E}}(Y_W(R_g u, x), x_0)Y_W(v, x) = Y_{\mathcal{E}}(Y_W(u, \phi(g)x), x_0)Y_W(v, x).$$

As it was mentioned in the proof of Proposition 2.13,

$$Y_{\mathcal{E}}(Y_W(u, \phi(g)x), x_0)Y_W(v, x) \in \mathcal{E}(W)[[x_0]] \quad \text{for all but finitely many } g \in \Gamma.$$

Thus

$$Y_W(Y(R_g u, x_0)v, x) \in \mathcal{E}(W)[[x_0]] \quad \text{for all but finitely many } g \in \Gamma.$$

Since Y_W is injective, the second assertion follows immediately. ∎

3 Relation between V-modules and $V^{\otimes N}$-quasi modules

In this section, we prove that for any $\mathbb{Z}$-graded vertex algebra V and any positive integer N, V-module structures on a vector space W one-to-one correspond to quasimodule structures for $V^{\otimes N}$ viewed as a Γ-vertex algebra. In view of [Li4], this can be considered as a version of a theorem of Barron, Dong and Mason [BDM] in terms of quasimodules.

The following is a simple result which is useful in application:

Lemma 3.1. *Let V be a Γ-vertex algebra and let (W, Y_W) be a quasimodule for V viewed as a vertex algebra. Assume that S is a generating subset of V as a vertex algebra such that $\{Y_W(u, x) \mid u \in S\}$ is $\phi(\Gamma)$-local and*

$$Y_W(R_g(u), x) = Y_W(u, \phi(g)x) \quad \text{for } g \in \Gamma,\ u \in S. \tag{3.1}$$

Then (W, Y_W) is a V-quasimodule.

Proof. As (W, Y_W) is a quasimodule for V viewed as a vertex algebra, $\bar{V} = \{Y_W(v, x) \mid v \in V\}$ is a closed quasilocal subspace of $\mathcal{E}(W)$, containing 1_W, and Y_W is a vertex-algebra homomorphism from V to $(\bar{V}, 1_W, Y_{\mathcal{E}})$. Set $\bar{S} = \{Y_W(u, x) \mid u \in S\} \subset \bar{V}$. Since S generates V as a vertex algebra, $\bar{S}$ generates $\bar{V}$ as a vertex algebra. With $\bar{S}$ being $\phi(\Gamma)$-local, from Theorem 2.9, $\bar{V}$ is $\phi(\Gamma)$-local. Set

$$K = \{v \in V \mid Y_W(R_g v, x) = Y_W(v, \phi(g)x) \quad \text{for } g \in \Gamma\}.$$

It remains to to prove $K = V$. Clearly, $\{\mathbf{1}\} \cup S \subset K$. Let $u, v \in K,\ g \in \Gamma,\ w \in W$. There exist nonzero polynomials $f(x_0, x)$ and $g(x_0, x)$ such that

$$\begin{aligned}
&f(x_0, x)Y_W(Y(R_g u, \phi(g)^{-1}x_0)R_g v, x)w = f(x_0, x)Y_W(R_g u, \phi(g)^{-1}x_0 + x)Y_W(R_g v, x)w,\\
&g(x_0, x)Y_W(u, x_0 + \phi(g)x)Y_W(v, \phi(g)x)w = g(x_0, x)Y_W(Y(u, x_0)v, \phi(g)x)w.
\end{aligned}$$

Then

$$\begin{aligned}
& f(x_0, x)g(x_0, x)Y_W(R_g Y(u, x_0)v, x)w\\
= \ & f(x_0, x)g(x_0, x)Y_W(Y(R_g u, \phi(g)^{-1}x_0)R_g v, x)w\\
= \ & f(x_0, x)g(x_0, x)Y_W(R_g u, \phi(g)^{-1}x_0 + x)Y_W(R_g v, x)w\\
= \ & f(x_0, x)g(x_0, x)Y_W(u, x_0 + \phi(g)x)Y_W(v, \phi(g)x)w\\
= \ & f(x_0, x)g(x_0, x)Y_W(Y(u, x_0)v, \phi(g)x)w,
\end{aligned}$$

from which we obtain

$$Y_W(R_g Y(u, x_0)v, x)w = Y_W(Y(u, x_0)v, \phi(g)x)w.$$

It follows that K is closed. As S generates V, we must have $K = V$. $\square$

Let V be a $\mathbb{Z}$-graded vertex algebra and let N be a fixed positive integer. From [FHL], we have a tensor product vertex algebra $V^{\otimes N}$, which is naturally $\mathbb{Z}$-graded. Let σ be the permutation automorphism of $V^{\otimes N}$ defined by

$$\sigma(v^1 \otimes \cdots \otimes v^{N-1} \otimes v^N) = v^2 \otimes \cdots \otimes v^N \otimes v^1$$

for $v^1, \ldots, v^N \in V$. Set

$$\Gamma = \langle \sigma \rangle \subset \mathrm{Aut}(V^{\otimes N}).$$

Let $\phi : \Gamma \to \mathbb{C}^\times$ be the group homomorphism defined by $\phi(\sigma) = \exp(2\pi i/N)$. Denote by Γ_N the group of nth roots of unity:

$$\Gamma_N = \{\exp(2\pi i r/N) \mid r = 0, \ldots, N-1\} \subset \mathbb{C}^\times.$$

We identify V as a vertex subalgebra of $V^{\otimes N}$ through the map

$$v \mapsto v \otimes \mathbf{1} \otimes \cdots \otimes \mathbf{1}.$$

Then $\sigma^j(V),\ j = 0, \ldots, N-1$, are mutually commuting graded vertex subalgebras and $V^{\otimes N} = \otimes_{j=0}^{N-1} \sigma^j(V)$. For $g \in \Gamma$, we define

$$R_g a = \phi(g)^{-L(0)} g(a) \quad \text{for } g \in \Gamma,\ a \in V^{\otimes N}.$$

In view of Example 2.3, $V^{\otimes N}$ becomes a Γ-vertex algebra. With all these notations we have:

Theorem 3.2. *Let $(W, \overline{Y}_W)$ be a $V^{\otimes N}$-quasimodule. Denote by Y_W the restriction map of $\overline{Y}_W$ to V. Then (W, Y_W) is a V-module. On the other hand, for any V-module (W, Y_W), the linear map Y_W from V to $\mathrm{Hom}(W, W((x)))$ can be extended to a linear map $\overline{Y}_W$ from $V^{\otimes N}$ to $\mathrm{Hom}(W, W((x)))$ such that $(W, \overline{Y}_W)$ is a $V^{\otimes N}$-quasimodule. Furthermore, such an extension is unique.*

Proof. Let $(W, \overline{Y}_W)$ be a $V^{\otimes N}$-quasimodule. Then (W, Y_W) is a quasimodule for V (a vertex algebra). For $u, v \in V,\ g \in \Gamma$ with $g \neq 1$, we have

$$Y(g(u), x)v \in (V^{\otimes N})[[x]].$$

Using this property and Proposition 2.13 we get

$$\begin{aligned}
& [\overline{Y}_W(u, x_1), \overline{Y}_W(v, x_2)] \\
= & \sum_{g \in \Gamma} \mathrm{Res}_{x_0} x_1^{-1} \delta\left(\frac{\phi(g)x_2 + x_0}{x_1}\right) \overline{Y}_W(Y(\phi(g)^{-L(0)} g(u), \phi(g)^{-1} x_0)v, x_2) \\
= & \mathrm{Res}_{x_0} x_1^{-1} \delta\left(\frac{x_2 + x_0}{x_1}\right) \overline{Y}_W(Y(u, x_0)v, x_2).
\end{aligned}$$

It follows that

$$(x_1 - x_2)^k [\overline{Y}_W(u, x_1), \overline{Y}_W(v, x_2)] = 0$$

for any nonnegative integer k with $x^kY(u,x)v \in V[[x]]$. In view of Lemma 2.1, (W, Y_W) is a V-module.

For the other direction, first we prove the uniqueness. Suppose that a V-module structure Y_W on W is extended to a $V^{\otimes N}$-quasimodule structure $\overline{Y}_W$. By ([Li3], Proposition 5.4), $\overline{Y}_W$ is a vertex-algebra homomorphism. For $g \in \Gamma,\ v \in V$, we have

$$\overline{Y}_W(gv, x) = \overline{Y}_W(\phi(g)^{L(0)}v, \phi(g)x) = Y_W(\phi(g)^{L(0)}v, \phi(g)x). \tag{3.2}$$

Since $g(V)$, $g \in \Gamma$, generate $V^{\otimes N}$ as a vertex algebra, the uniqueness is clear.

Now it remains to establish the existence. For $g \in \Gamma$, set

$$U[g] = \{Y_W(\phi(g)^{L(0)}v, \phi(g)x) \mid v \in V\} \subset \mathrm{Hom}(W, W((x))) = \mathcal{E}(W).$$

Furthermore, set

$$U = \sum_{g\in\Gamma} U[g] \subset \mathcal{E}(W).$$

For $u, v \in V,\ \alpha, \beta \in \mathbb{C}^{\times}$, there exists a nonnegative integer k such that

$$(x_1 - x_2)^k [Y_W(\alpha^{L(0)}u, x_1), Y_W(\beta^{L(0)}v, x_2)] = 0.$$

Consequently,

$$(x_1 - \alpha^{-1}\beta x_2)^k [Y_W(\alpha^{L(0)}u, \alpha x_1), Y_W(\beta^{L(0)}v, \beta x_2)] = 0. \tag{3.3}$$

It follows that U is Γ_N-local. By Theorem 2.9, U generates a Γ_N-vertex algebra $\langle U\rangle_{\Gamma_N}$ with W as a Γ_N-quasimodule. Furthermore, for every $g \in \Gamma$, from (3.3) $U[g]$ is local, so from [Li1], $U[g]$ generates a vertex algebra with W as a module, where for $u, v \in V$,

$$\begin{aligned}
& Y_{\mathcal{E}}(Y_W(\phi(g)^{L(0)}u, \phi(g)x), x_0) Y_W(\phi(g)^{L(0)}v, \phi(g)x) \\
= \ & \mathrm{Res}_{x_1} x_0^{-1}\delta\left(\frac{x_1 - x}{x_0}\right) Y_W(\phi(g)^{L(0)}u, \phi(g)x_1) Y_W(\phi(g)^{L(0)}v, \phi(g)x) \\
& - \mathrm{Res}_{x_1} x_0^{-1}\delta\left(\frac{x - x_1}{-x_0}\right) Y_W(\phi(g)^{L(0)}v, \phi(g)x) Y_W(\phi(g)^{L(0)}u, \phi(g)x_1) \\
= \ & \mathrm{Res}_{x_1} x_1^{-1}\delta\left(\frac{x + x_0}{x_1}\right) Y_W(Y(\phi(g)^{L(0)}u, \phi(g)x_0)\phi(g)^{L(0)}v, \phi(g)x) \\
= \ & Y_W(\phi(g)^{L(0)}Y(u, x_0)v, \phi(g)x).
\end{aligned}$$

This shows that the linear map

$$f_g : V \to U[g] \subset \langle U\rangle_{\Gamma}, \quad v \mapsto Y_W(\phi(g)^{L(0)}v, \phi(g)x)$$

is a vertex-algebra homomorphism. Furthermore, if $g \neq h$, with the relation (3.3) it follows from ([Li3], Proposition 4.8) that for $a(x) \in U[g],\ b(x) \in U[h]$,

$$[Y_{\mathcal{E}}(a(x), x_1), Y_{\mathcal{E}}(b(x), x_2)] = 0.$$

Then vertex-algebra homomorphisms f_g ($g \in \Gamma$) give rise to a vertex-algebra homomorphism $\overline{Y}_W$ from $V^{\otimes N}$ to $\langle U \rangle_{\Gamma_N}$. Consequently, W is a quasimodule for $V^{\otimes N}$ viewed as a vertex algebra, where for $v \in V,\ g \in \Gamma$,

$$\overline{Y}_W(g(v), x) = Y_W(\phi(g)^{L(0)}v, \phi(g)x).$$

For $g, h \in \Gamma,\ v \in V$, we have

$$\begin{aligned}
\overline{Y}_W(\phi(g)^{-L(0)}g(h(v)), x) &= \overline{Y}_W(\phi(g)^{-L(0)}gh(v), x) \\
&= Y_W(\phi(h)^{L(0)}v, \phi(gh)x) \\
&= \overline{Y}_W(h(v), \phi(g)x).
\end{aligned}$$

By Lemma 3.1, W is a quasimodule for $V^{\otimes N}$ viewed as a Γ-vertex algebra. $\square$

4 Certain generalizations of twisted affine Lie algebras

In this section we extend the results of Golenishcheva-Kutuzova and Kac (see [GKK]) on a certain generalization of the construction of twisted affine Lie algebras. We show that restricted modules for such generalized twisted affine Lie algebras are quasimodules for vertex algebras associated with (untwisted) affine Lie algebras. We also formulate a notion of Γ-conformal Lie algebra, which extends the notion of Γ-conformal Lie algebra of [GKK].

First we prove the following simple result:

Lemma 4.1. *Let K be a Lie algebra equipped with a (possibly zero) symmetric invariant bilinear form $\langle \cdot, \cdot \rangle$. Assume that a group Γ acts on K by automorphisms preserving the bilinear form such that for any $u, v \in K$,*

$$[gu, v] = 0, \quad \langle gu, v \rangle = 0 \quad \text{for all but finitely many } g \in \Gamma.$$

Define a new multiplicative operation $[\cdot, \cdot]_\Gamma$ on K by

$$[u, v]_\Gamma = \sum_{g \in \Gamma} [gu, v] \tag{4.1}$$

for $u, v \in K$. Then the subspace linearly spanned by the vectors $gu - u$ for $g \in \Gamma,\ u \in K$ is a two-sided ideal of the new nonassociative algebra and the quotient algebra which we denote by K/Γ is a Lie algebra. Define a bilinear form $\langle \cdot, \cdot \rangle_\Gamma$ on K by

$$\langle u, v \rangle_\Gamma = \sum_{g \in \Gamma} \langle gu, v \rangle \quad \text{for } u, v \in K.$$

Then $\langle \cdot, \cdot \rangle_\Gamma$ naturally gives rise to a symmetric invariant bilinear form on K/Γ.

Proof. Let $g \in \Gamma,\ u, v, w \in K$. We have

$$[gu - u, v]_\Gamma = \sum_{h\in\Gamma}[hgu, v] - \sum_{h\in\Gamma}[hu, v] = \sum_{k\in\Gamma}[ku, v] - \sum_{h\in\Gamma}[hu, v] = 0.$$

Using the assumption that Γ acts on K by automorphisms, we have

$$[v, u]_\Gamma = \sum_{g\in\Gamma}[g^{-1}v, u] = \sum_{g\in\Gamma} g^{-1}[v, gu] = -\sum_{g\in\Gamma} g^{-1}[gu, v],$$

from which we get

$$[u, v]_\Gamma + [v, u]_\Gamma = \sum_{g\in\Gamma}([gu, v] - g^{-1}[gu, v]).$$

Furthermore, we have

$$\begin{aligned} & [u, [v, w]_\Gamma]_\Gamma - [v, [u, w]_\Gamma]_\Gamma - [[u, v]_\Gamma, w]_\Gamma \\ = & \sum_{g,h\in\Gamma}[gu, [hv, w]] - \sum_{g,h\in\Gamma}[hv, [gu, w]] - \sum_{g,h\in\Gamma}[[hgu, hv], w] = 0. \end{aligned}$$

From these the first assertion follows.

For the second assertion, for $g \in \Gamma,\ u, v \in K$, we have

$$\langle gu - u, v\rangle_\Gamma = \sum_{h\in\Gamma}\langle h(gu - u), v\rangle = \sum_{h\in\Gamma}\langle hgu, v\rangle - \sum_{h\in\Gamma}\langle hu, v\rangle = 0.$$

The bilinear form $\langle\cdot,\cdot\rangle_\Gamma$ is symmetric as

$$\langle v, u\rangle_\Gamma = \sum_{g\in\Gamma}\langle gv, u\rangle = \sum_{g\in\Gamma}\langle g^{-1}u, v\rangle = \langle u, v\rangle_\Gamma.$$

For $h \in \Gamma,\ u, v, w \in K$, we have

$$\langle hu, hv\rangle_\Gamma = \sum_{g\in\Gamma}\langle ghu, hv\rangle = \sum_{g\in\Gamma}\langle h^{-1}ghu, v\rangle = \langle u, v\rangle_\Gamma$$

and

$$\langle [u, v]_\Gamma, w\rangle_\Gamma = \sum_{g,h\in\Gamma}\langle g[hu, v], w\rangle = \sum_{g,h\in\Gamma}\langle [ghu, gv], w\rangle = \sum_{g,h\in\Gamma}\langle ghu, [gv, w]\rangle = \langle u, [v, w]_\Gamma\rangle_\Gamma.$$

From these the second assertion follows. $\square$

Remark 4.2. Let Γ be a finite group acting on a Lie algebra K by automorphisms. Then the assumption in Lemma 4.1 automatically holds. On the one hand, we have the Γ-invariant Lie subalgebra K^Γ (the set of Γ-fixed points) and on the other hand, we have the Lie algebra K/Γ. It is straightforward to show that the linear map $\psi : K \to K^\Gamma$ defined by $\psi(u) = \sum_{g\in\Gamma} gu$ gives rise to a Lie algebra isomorphism from K/Γ onto K^Γ.

Remark 4.3. In [GKK], Golenishcheva-Kutuzova and Kac studied a notion of Γ-conformal algebra and they proved that a Γ-conformal algebra structure on a vector space $\mathfrak{g}$ exactly amounts to a Lie algebra structure together with a group action of Γ on $\mathfrak{g}$ by automorphisms such that for any $u, v \in \mathfrak{g}$, $[gu, v] = 0$ for all but finitely many $g \in \Gamma$. Furthermore, a loop-like (or current-like) Lie algebra was associated to every Γ-conformal algebra together with a group homomorphism from Γ to $\mathbb{C}^\times$.

The following proposition extends a result of [GKK] with a central extension included (with a different proof):

Proposition 4.4. *Let $\mathfrak{g}$ be a (possibly infinite-dimensional) Lie algebra equipped with a symmetric invariant bilinear form $\langle \cdot, \cdot \rangle$, let Γ be a subgroup of $\mathrm{Aut}(\mathfrak{g}, \langle \cdot, \cdot \rangle)$ and let ϕ be any group homomorphism from Γ to $\mathbb{C}^\times$. Assume that for $a, b \in \mathfrak{g}$,*

$$[ga, b] = 0 \quad \textit{and} \quad \langle ga, b \rangle = 0 \quad \textit{for all but finitely many } g \in \Gamma. \tag{4.2}$$

Define a bilinear multiplicative operation $[\cdot, \cdot]_\Gamma$ on the vector space $\mathfrak{g} \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}\mathbf{k}$ by

$$[a \otimes t^m + \alpha\mathbf{k}, b \otimes t^n + \beta\mathbf{k}]_\Gamma = \sum_{g \in \Gamma} \phi(g)^m \left([ga, b] \otimes t^{m+n} + m \langle ga, b \rangle \delta_{m+n,0} \mathbf{k} \right) \tag{4.3}$$

for $a, b \in \mathfrak{g},\ m, n \in \mathbb{Z},\ \alpha, \beta \in \mathbb{C}$. Then the subspace linearly spanned by the elements

$$\phi(g)^m (ga \otimes t^m) - (a \otimes t^m) \quad \textit{for } g \in \Gamma,\ a \in \mathfrak{g},\ m \in \mathbb{Z} \tag{4.4}$$

is a two-sided ideal of the nonassociative algebra and the quotient algebra which we denote by $\hat{\mathfrak{g}}[\Gamma]$ is a Lie algebra.

Proof. Associated to the pair $(\mathfrak{g}, \langle \cdot, \cdot \rangle)$, we have the (untwisted) affine Lie algebra

$$\hat{\mathfrak{g}} = \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}\mathbf{k}, \tag{4.5}$$

where $\mathbf{k}$ is central, and for $a, b \in \mathfrak{g},\ m, n \in \mathbb{Z}$,

$$[a \otimes t^m, b \otimes t^n] = [a, b] \otimes t^{m+n} + m\delta_{m+n,0} \langle a, b \rangle \mathbf{k}. \tag{4.6}$$

Let Γ act on $\hat{\mathfrak{g}}$ by

$$g(a \otimes t^m + \beta\mathbf{k}) = \phi(g)^m (ga \otimes t^m) + \beta\mathbf{k}$$

for $g \in \Gamma,\ a \in \mathfrak{g},\ m \in \mathbb{Z},\ \beta \in \mathbb{C}$. It is straightforward to see that Γ acts on $\hat{\mathfrak{g}}$ by automorphisms. Furthermore, for any $a, b \in \mathfrak{g},\ m, n \in \mathbb{Z},\ \alpha, \beta \in \mathbb{C}$, we have

$$[g(a \otimes t^m + \alpha\mathbf{k}), b \otimes t^n + \beta\mathbf{k}] = \phi(g)^m \left([ga, b] \otimes t^{m+n} + m\delta_{m+n,0} \langle ga, b \rangle \mathbf{k} \right) = 0$$

for all but finitely many $g \in \Gamma$. Now it follows immediately from Lemma 4.1 with $K = \hat{\mathfrak{g}}$. $\square$

Remark 4.5. Let $\mathfrak{g}, \langle \cdot, \cdot \rangle$ be given as in Proposition 4.4, and let σ be an order N automorphism of $\mathfrak{g}$, preserving the bilinear form $\langle \cdot, \cdot \rangle$. Extend σ to an automorphism of Lie algebra $\hat{\mathfrak{g}}$ by

$$\sigma(u \otimes t^m + \alpha \mathbf{k}) = \exp(-2m\pi i/N)(\sigma(u) \otimes t^m) + \alpha \mathbf{k}$$

for $u \in \mathfrak{g},\ m \in \mathbb{Z},\ \alpha \in \mathbb{C}$. The twisted affine Lie algebra $\hat{\mathfrak{g}}[\sigma]$ (see [K1]) can be realized as the σ-fixed point subalgebra of $\hat{\mathfrak{g}}$. Set $\Gamma = \langle \sigma \rangle \subset \mathrm{Aut}(\mathfrak{g}, \langle \cdot, \cdot \rangle)$ and let ϕ be the group embedding of Γ into $\mathbb{C}^{\times}$ defined by $\phi(\sigma^n) = \exp(-2n\pi i/N)$. We let Γ act on $\hat{\mathfrak{g}}$ as in the proof of Proposition 4.4. Clearly, $\hat{\mathfrak{g}}[\sigma]$ is also the Γ-invariant subalgebra. In view of this and Remark 4.2, Lie algebras $\hat{\mathfrak{g}}[\Gamma]$ are generalizations of twisted affine Lie algebras.

Remark 4.6. Let $\mathfrak{g}, \langle \cdot, \cdot \rangle, \Gamma,\ \phi$ be given as in Proposition 4.4. Set $H = \ker \phi \subset \Gamma$, a normal subgroup. In view of Lemma 4.1, we have a Lie algebra $\mathfrak{g}/H$ equipped with a symmetric invariant bilinear form $\langle \cdot, \cdot \rangle_H$. Then we have the (untwisted) affine Lie algebra $\widehat{\mathfrak{g}/H}$. On the other hand, Γ/H naturally acts on the Lie algebra $\mathfrak{g}/H$ by automorphisms and ϕ reduces to a group embedding of Γ/H into $\mathbb{C}^{\times}$. In view of Proposition 4.4, we have a Lie algebra $(\widehat{\mathfrak{g}/H})[\Gamma/H]$.

We have:

Proposition 4.7. *Let $\mathfrak{g}, \langle \cdot, \cdot \rangle, \Gamma$ and ϕ be given as in Proposition 4.4 and set $H = \ker \phi \subset \Gamma$. The Lie algebra $\hat{\mathfrak{g}}[\Gamma]$ is isomorphic to the Lie algebra $(\widehat{\mathfrak{g}/H})[\Gamma/H]$.*

Proof. It is straightforward. □

Remark 4.8. We here review the $\mathbb{Z}$-graded vertex algebras associated (untwisted) affine Lie algebras. Let $\mathfrak{g}$ be a Lie algebra equipped with a symmetric invariant bilinear form $\langle \cdot, \cdot \rangle$ and let $\hat{\mathfrak{g}}$ be the associated affine Lie algebra. For $a \in \mathfrak{g}$, form the generating function

$$a(x) = \sum_{n \in \mathbb{Z}} a(n) x^{-n-1},$$

where $a(n)$ is an alternative notation for $a \otimes t^n$. Lie algebra $\hat{\mathfrak{g}}$ is naturally $\mathbb{Z}$-graded $\hat{\mathfrak{g}} = \coprod_{n \in \mathbb{Z}} \hat{\mathfrak{g}}_{(n)}$, where

$$\hat{\mathfrak{g}}_{(0)} = \mathfrak{g} \oplus \mathbb{C}\mathbf{k}, \quad \hat{\mathfrak{g}}_{(n)} = \mathfrak{g} \otimes t^{-n} \quad \text{for } n \neq 0.$$

Set

$$\hat{\mathfrak{g}}_{(\leq 0)} = \mathfrak{g} \otimes \mathbb{C}[t] \oplus \mathbb{C}\mathbf{k}, \quad \hat{\mathfrak{g}}_{(+)} = \mathfrak{g} \otimes t^{-1}\mathbb{C}[t^{-1}].$$

Let ℓ be a complex number and let $\mathbb{C}_\ell = \mathbb{C}$ be the 1-dimensional $\hat{\mathfrak{g}}_{(\leq 0)}$-module with $\mathfrak{g} \otimes \mathbb{C}[t]$ acting trivially and with $\mathbf{k}$ acting as scalar ℓ. Form the induced module

$$V_{\hat{\mathfrak{g}}}(\ell, 0) = U(\hat{\mathfrak{g}}) \otimes_{U(\hat{\mathfrak{g}}_{(\leq 0)})} \mathbb{C}_\ell,$$

which is naturally an $\mathbb{N}$-graded $\hat{\mathfrak{g}}$-module (of level ℓ). Set $\mathbf{1} = 1 \otimes 1$ and identify $\mathfrak{g}$ as a subspace through the map $a \mapsto a(-1)\mathbf{1}$. In fact, $\mathfrak{g}$ is exactly the degree-one subspace. It

was known (cf. [FZ], [Lia], [LL]) that there exists a (unique) vertex algebra structure on $V_{\hat{\mathfrak{g}}}(\ell,0)$ with $\mathbf{1}$ as the vacuum vector and with $Y(a,x)=a(x)$ for $a\in\mathfrak{g}$. Furthermore, vertex algebra $V_{\hat{\mathfrak{g}}}(\ell,0)$ satisfies the following universal property (cf. [P]): Let V be a vertex algebra and let f be a linear map from $\mathfrak{g}$ to V such that for $a,b\in\mathfrak{g}$,

$$f(a)_0f(b)=f([a,b]),\qquad f(a)_1f(b)=\ell\langle a,b\rangle\mathbf{1},\quad\text{and}\quad f(a)_nf(b)=0\quad\text{for } n\geq 2.$$

Then f can be extended (uniquely) to a vertex-algebra homomorphism from $V_{\hat{\mathfrak{g}}}(\ell,0)$ to V. Let $\sigma\in\mathrm{Aut}(\mathfrak{g},\langle\cdot,\cdot\rangle)$, that is, σ is an automorphism of Lie algebra $\mathfrak{g}$, which preserves the bilinear form. Then σ extends canonically to an automorphism of the $\mathbb{Z}$-graded vertex algebra $V_{\hat{\mathfrak{g}}}(\ell,0)$. For any subgroup Γ of $\mathrm{Aut}(\mathfrak{g},\langle\cdot,\cdot\rangle)$ and any group homomorphism $\phi:\Gamma\to\mathbb{C}^\times$, $V_{\hat{\mathfrak{g}}}(\ell,0)$ is a Γ-vertex algebra with $R_g=\phi(g)^{-L(0)}g$ for $g\in\Gamma$.

The following is the main result of this section:

Theorem 4.9. *Let $\mathfrak{g}$ be a Lie algebra equipped with a symmetric invariant bilinear form $\langle\cdot,\cdot\rangle$. Let Γ be a subgroup of $\mathrm{Aut}(\mathfrak{g},\langle\cdot,\cdot\rangle)$ such that for $a,b\in\mathfrak{g}$,*

$$[ga,b]=0\quad\text{and}\quad\langle ga,b\rangle=0\quad\text{for all but finitely many } g\in\Gamma.$$

Let ϕ be any group homomorphism from Γ to $\mathbb{C}^\times$ and set $H=\ker\phi\subset\Gamma$. Then any restricted module W of level ℓ for Lie algebra $\hat{\mathfrak{g}}[\Gamma]$ is a quasimodule for $V_{\widehat{\mathfrak{g}/H}}(\ell,0)$ viewed as a Γ-vertex algebra with $Y_W(\bar{a},x)=a_W(x)$ for $a\in\mathfrak{g}$, where $\bar{a}$ denotes the image of a in $\mathfrak{g}/H$ under the natural quotient map. On the other hand, any quasimodule (W,Y_W) for $V_{\widehat{(\mathfrak{g}/H)}}(\ell,0)$ viewed as a Γ-vertex algebra is a restricted module of level ℓ for Lie algebra $\hat{\mathfrak{g}}[\Gamma]$ with $a_W(x)=Y_W(\bar{a},x)$ for $a\in\mathfrak{g}$.

Proof. Let W be a restricted $\hat{\mathfrak{g}}[\Gamma]$-module of level ℓ. For $a\in\mathfrak{g}$, form the generating function

$$a_W(x)=\sum_{n\in\mathbb{Z}}\overline{a(n)}x^{-n-1}\in\mathcal{E}(W),$$

where $\overline{a(n)}$ denotes the operator on W, associated to the image of $a\otimes t^n$ in $\hat{\mathfrak{g}}[\Gamma]$. Set

$$U=\{a_W(x)\mid a\in\mathfrak{g}\}\subset\mathcal{E}(W).$$

For $a,b\in\mathfrak{g}$, we have

$$[a_W(x_1),b_W(x_2)]=\sum_{g\in\Gamma}[ga,b]_W(x_2)x_1^{-1}\delta\left(\frac{\phi(g)x_2}{x_1}\right)+\ell\langle ga,b\rangle\frac{\partial}{\partial x_2}x_1^{-1}\delta\left(\frac{\phi(g)x_2}{x_1}\right).\quad(4.7)$$

It follows that U is $\phi(\Gamma)$-local. By Theorem 2.9 U generates a $\phi(\Gamma)$-vertex algebra $\langle U\rangle_{\phi(\Gamma)}$ with W as a faithful quasimodule where $Y_W(\alpha(x),x_0)=\alpha(x_0)$ for $\alpha(x)\in\langle U\rangle_{\phi(\Gamma)}$. As for $h\in H=\ker\phi$,

$$(ha)_W(x)=\phi(h)a_W(\phi(h)x)=a_W(x),$$

we have a linear map from $\mathfrak{g}/H$ into $\langle U\rangle_{\phi(\Gamma)}$, sending $\bar{a}$ to $a_W(x)$ for $a\in\mathfrak{g}$. Furthermore, for $a,b\in\mathfrak{g}$ we have

$$\begin{aligned}
&[Y_W(a_W(x),x_1),Y_W(b_W(x),x_2)]\\
=\ &[a_W(x_1),b_W(x_2)]\\
=\ &\sum_{g\in\Gamma}[ga,b]_W(x_2)x_1^{-1}\delta\left(\frac{\phi(g)x_2}{x_1}\right)+\ell\langle ga,b\rangle\frac{\partial}{\partial x_2}x_1^{-1}\delta\left(\frac{\phi(g)x_2}{x_1}\right)\\
=\ &\sum_{g\in\Gamma}Y_W([ga,b],x_2)x_1^{-1}\delta\left(\frac{\phi(g)x_2}{x_1}\right)+\ell\langle ga,b\rangle\frac{\partial}{\partial x_2}x_1^{-1}\delta\left(\frac{\phi(g)x_2}{x_1}\right).
\end{aligned}$$

Comparing this with Proposition 2.13, we obtain

$$\begin{aligned}
&a_W(x)_0b_W(x)=\sum_{h\in H}[ha,b]_W(x),\quad a_W(x)_1b_W(x)=\ell\sum_{h\in H}\langle ha,b\rangle 1_W,\\
&a_W(x)_nb_W(x)=0\quad\text{for } n\geq 2.
\end{aligned}$$

In view of Remark 4.8, there exists a (unique) vertex-algebra homomorphism from $V_{\widehat{\mathfrak{g}/H}}(\ell,0)$ to $\langle U\rangle_{\phi(\Gamma)}$, sending $\bar{a}$ to $a_W(x)$ for $a\in\mathfrak{g}$. It follows from Lemma 3.1 that W is a quasimodule for $V_{\widehat{(\mathfrak{g}/H)}}(\ell,0)$ viewed as a vertex algebra with $Y_W(\bar{a},x)=a_W(x)$ for $a\in\mathfrak{g}$. For $g\in\Gamma,\ a\in\mathfrak{g}$, we have

$$Y_W(R_ga,x)=\phi(g)^{-1}Y_W(ga,x)=\phi(g)^{-1}(ga)_W(x)=a_W(\phi(g)x)=Y_W(a,\phi(g)x).$$

Now it follows from Lemma 3.1 that W is a quasimodule for $V_{\widehat{(\mathfrak{g}/H)}}(\ell,0)$ viewed as a Γ-vertex algebra.

On the other hand, let (W,Y_W) be a quasimodule for $V_{\widehat{(\mathfrak{g}/H)}}(\ell,0)$ viewed as a Γ/H-vertex algebra. For $a\in\mathfrak{g}$, set $a_W(x)=Y_W(\bar{a},x)$. For $g\in\Gamma,\ a\in\mathfrak{g}$, we have

$$(ga)_W(x)=Y_W(\overline{ga},x)=Y_W(\bar{g}\bar{a},x)=\bar{\phi}(\bar{g})Y_W(\bar{a},\bar{\phi}(\bar{g})x)=\phi(g)a_W(\phi(g)x).$$

Notice that for $u,v\in\mathfrak{g}$, with $\bar{u},\bar{v}\in V_{\widehat{(\mathfrak{g}/H)}}(\ell,0)$, we have

$$\bar{u}_0\bar{v}=[\bar{u},\bar{v}]=\sum_{h\in H}\overline{[hu,v]},\ \ \bar{u}_1\bar{v}=\ell\langle\bar{u},\bar{v}\rangle\mathbf{1}=\sum_{h\in H}\ell\langle hu,v\rangle\mathbf{1},\ \ \bar{u}_n\bar{v}=0\quad\text{for } n\geq 2.$$

For $a, b \in \mathfrak{g}$, using Proposition 2.13 we have

$$
\begin{aligned}
& [a_W(x_1), b_W(x_2)] \\
= \; & [Y_W(\bar{a}, x_1), Y_W(\bar{b}, x_2)] \\
= \; & \sum_{\bar{g} \in \Gamma/H} \mathrm{Res}_{x_0} x_1^{-1} \delta\left(\frac{\bar{\phi}(\bar{g}) x_2 + x_0}{x_1}\right) Y_W(Y(R_{\bar{g}} \bar{a}, \bar{\phi}(\bar{g})^{-1} x_0) \bar{b}, x_2) \\
= \; & \sum_{\bar{g} \in \Gamma/H} Y_W([\bar{g}\bar{a}, \bar{b}], x_2) x_1^{-1} \delta\left(\frac{\bar{\phi}(\bar{g}) x_2}{x_1}\right) + \ell \langle \bar{g}\bar{a}, \bar{b} \rangle \frac{\partial}{\partial x_2} x_1^{-1} \delta\left(\frac{\bar{\phi}(\bar{g}) x_2}{x_1}\right) \\
= \; & \sum_{g \in \Gamma} Y_W(\overline{[ga, b]}, x_2) x_1^{-1} \delta\left(\frac{\phi(g) x_2}{x_1}\right) + \ell \langle ga, b \rangle \frac{\partial}{\partial x_2} x_1^{-1} \delta\left(\frac{\phi(g) x_2}{x_1}\right) \\
= \; & \sum_{g \in \Gamma} [ga, b]_W(x_2) x_1^{-1} \delta\left(\frac{\phi(g) x_2}{x_1}\right) + \ell \langle ga, b \rangle \frac{\partial}{\partial x_2} x_1^{-1} \delta\left(\frac{\phi(g) x_2}{x_1}\right).
\end{aligned}
$$

It follows that W is a restricted $\hat{\mathfrak{g}}[\Gamma]$-module of level ℓ with $u_W(x) = Y_W(\bar{u}, x)$ for $u \in \mathfrak{g}$. $\square$

Example 4.10. Let Γ be a group as before. We define an associative algebra gl_Γ with a basis $\{E_{\alpha,\beta} \mid \alpha, \beta \in \Gamma\}$ such that

$$E_{\alpha,\beta} \cdot E_{\lambda,\mu} = \delta_{\beta,\lambda} E_{\alpha,\mu} \quad \text{for } \alpha, \beta, \lambda, \mu \in \Gamma. \tag{4.8}$$

Equip gl_Γ with a bilinear form $\langle \cdot, \cdot \rangle$ defined by

$$\langle E_{\alpha,\beta}, E_{\lambda,\mu} \rangle = \delta_{\alpha,\mu} \delta_{\beta,\lambda} \quad \text{for } \alpha, \beta, \lambda, \mu \in \Gamma. \tag{4.9}$$

Clearly, this form is nondegenerate, symmetric and associative (invariant). For any associative algebra A equipped with a nondegenerate symmetric invariant bilinear form $\langle \cdot, \cdot \rangle$, the tensor product associative algebra $A \otimes gl_\Gamma$ has a nondegenerate symmetric invariant bilinear form with

$$\langle a \otimes E_{\alpha,\beta}, b \otimes E_{\lambda,\mu} \rangle = \langle a, b \rangle \langle E_{\alpha,\beta}, E_{\lambda,\mu} \rangle = \delta_{\alpha,\mu} \delta_{\beta,\lambda} \langle a, b \rangle \tag{4.10}$$

for $a, b \in A,\ \alpha, \beta, \lambda, \mu \in \Gamma$. The bilinear form is still invariant with $A \otimes gl_\Gamma$ viewed as a Lie algebra. Then we have a (generalized) affine Lie algebra

$$\widehat{A \otimes gl}_\Gamma = (A \otimes gl_\Gamma) \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}\mathbf{k}. \tag{4.11}$$

Let Γ act on $A \otimes gl_\Gamma$ by

$$T_g(a \otimes E_{\alpha,\beta}) = a \otimes E_{g\alpha, g\beta} \quad \text{for } g, \alpha, \beta \in \Gamma,\ a \in A. \tag{4.12}$$

Clearly, this defines an action of Γ on $A \otimes gl_\Gamma$ by automorphisms and for $g \in \Gamma$, T_g preserves the bilinear form. Let ϕ be any group homomorphism from Γ to $\mathbb{C}^\times$. Set

$$R_g = \phi(g)^{-1} T_g \quad \text{for } g \in \Gamma. \tag{4.13}$$

It is clear that for any $u, v \in A \otimes gl_\Gamma$,

$$[R_g u, v] = 0 \quad \text{and} \quad \langle R_g u, v\rangle = 0 \quad \text{for all but finitely many } g \in \Gamma. \tag{4.14}$$

In view of Proposition 4.4 we have a Lie algebra $\widehat{A \otimes gl_\Gamma}[\Gamma]$. By Theorem 4.9, any restricted module W of level ℓ for $\widehat{A \otimes gl_\Gamma}[\Gamma]$ is naturally a quasimodule for some vertex algebra. Now, let $\Gamma = \mathbb{Z}^k$ be a free abelian group of rank k and let $\hbar = (h_1, \dots, h_k) \in \mathbb{R}^k$. Define a group homomorphism $\phi_\hbar$ from $\mathbb{Z}^k$ to $\mathbb{C}^\times$ by

$$\phi_\hbar(n_1, \dots, n_k) = e^{\pi i(h_1 n_1 + \cdots + h_k n_k)}. \tag{4.15}$$

We have a Lie algebra $\widehat{gl_{\mathbb{Z}^k}}[\mathbb{Z}^k]$. This is the Lie algebra $\hat{A}_\hbar$ (with central extension) in [GKL1.2] (cf. [GKK]).

Remark 4.11. Let V be a vertex operator algebra in the sense of [FLM] and [FHL], let Γ be a group of automorphisms of V and let $\phi : \Gamma \to \mathbb{C}^\times$ be a group homomorphism. View V as a Γ-vertex algebra as in Example 2.3. Let (W, Y_W) be a V-quasimodule. For $g \in \Gamma$, as $g(\omega) = \omega$, we have

$$R_g \omega = \phi(g)^{-L(0)} g(\omega) = \phi(g)^{-2}\omega,$$

recalling that ω is the conformal vector of V. Then

$$Y_W(\omega, x) = \phi(g)^2 Y_W(\omega, \phi(g)x).$$

If $\phi(g)$ is not a root of unity for some $g \in \Gamma$, then $Y_W(\omega, x) = 0$. Thus if V is simple with $\omega \neq 0$ and if $\phi(g)$ is not a root of unity for some $g \in \Gamma$, V does not have a nonzero quasimodule for V viewed as a Γ-vertex algebra. Nevertheless, as we show by examples in the next Remark, there are nontrivial quasimodules for V viewed just as a vertex algebra.

Remark 4.12. Consider the simple Lie algebra sl_{n+1} as a subalgebra of gl_∞ $(= gl_{\mathbb{Z}})$ in the obvious way. Let K be the linear span of $T_m sl_{n+1}$ for $m \in \mathbb{Z}$, that is, K is linearly spanned by

$$E_{i+m, j+m}, \quad E_{i+m,i+m} - E_{j+m,j+m}$$

for $1 \leq i \neq j \leq n+1$, $m \in \mathbb{Z}$. Then K is a Lie subalgebra of gl_∞ with $\mathbb{Z}$ as a group of automorphisms. Let ϕ be any injective group homomorphism from $\mathbb{Z}$ to $\mathbb{C}^\times$. We have a Lie algebra $\hat{K}[\mathbb{Z}]$. By Theorem 4.9, any restricted $\hat{K}[\mathbb{Z}]$-module of level $\ell \in \mathbb{C}$ is naturally a quasimodule for $V_{\hat{K}}(\ell, 0)$. With sl_{n+1} being a Lie subalgebra of K, $V_{\widehat{sl_{n+1}}}(\ell, 0)$ is naturally a vertex subalgebra of $V_{\hat{K}}(\ell, 0)$. Consequently, any restricted $\hat{K}[\mathbb{Z}]$-module of level $\ell \in \mathbb{C}$ is naturally a quasimodule for $V_{\widehat{sl_{n+1}}}(\ell, 0)$ (as a vertex algebra). More generally, for any finite-dimensional simple Lie algebra $\mathfrak{g}$, one can embed $\mathfrak{g}$ into gl_∞, we can obtain nontrivial $V_{\hat{\mathfrak{g}}}(\ell, 0)$-quasimodules.

Next, we extend the notion of Γ-conformal Lie algebra of [GKK]. First, recall that a *conformal Lie algebra* [K2], also known as a *vertex Lie algebra* [P] (cf. [DLM]), is a vector space C equipped with a linear operator T and a linear map

$$\begin{aligned} Y_- : \quad & C \to \mathrm{Hom}(C, x^{-1}C[x^{-1}]) \\ & a \mapsto Y_-(a,x) = \sum_{n \geq 0} a_n x^{-n-1} \end{aligned} \tag{4.16}$$

such that the following conditions hold for $a, b \in C$:

$$[T, Y_-(a,x)] = \frac{d}{dx} Y_-(a,x), \tag{4.17}$$

$$Y_-(a,x)b = \mathrm{Sing}\left(e^{xT} Y_-(b,-x)a\right), \tag{4.18}$$

$$[Y_-(a,x_1), Y_-(b,x_2)] = \mathrm{Sing}\left(Y_-(Y_-(a, x_1 - x_2)b, x_2)\right), \tag{4.19}$$

where Sing stands for the singular part.

Associated to a conformal Lie algebra C one has a Lie algebra $\mathcal{L}(C)$ (see [P]), where

$$\mathcal{L}(C) = (C \otimes \mathbb{C}[t,t^{-1}])/(T \otimes 1 + 1 \otimes d/dt)(C \otimes \mathbb{C}[t,t^{-1}]), \tag{4.20}$$

as a vector space, and for $a, b \in C,\ m, n \in \mathbb{Z}$,

$$[a \otimes t^m, b \otimes t^n] = \sum_{i \geq 0} \binom{m}{i} (a_i b) \otimes t^{m+n-i}. \tag{4.21}$$

Denote by ρ the natural quotient map from $C \otimes \mathbb{C}[t,t^{-1}]$ onto $\mathcal{L}(C)$. For $u \in C,\ n \in \mathbb{Z}$, set

$$u(n) = \rho(u \otimes t^n) \in \mathcal{L}(C)$$

and form the generating function

$$u(x) = \sum_{n \in \mathbb{Z}} u(n) x^{-n-1} \in \mathcal{L}(C)[[x, x^{-1}]].$$

Set

$$\mathcal{L}(C)^+ = \rho(C \otimes \mathbb{C}[t]), \qquad \mathcal{L}(C)^- = \rho(C \otimes t^{-1}\mathbb{C}[t^{-1}]).$$

Then $\mathcal{L}(C)^\pm$ are Lie subalgebras and we have

$$\mathcal{L}(C) = \mathcal{L}(C)^+ \oplus \mathcal{L}(C)^-.$$

Letting $\mathcal{L}(C)^+$ act trivially on $\mathbb{C}$, we form the induced module

$$V_C = U(\mathcal{L}(C)) \otimes_{U(\mathcal{L}(C)^+)} \mathbb{C}.$$

Set $\mathbf{1} = 1 \otimes 1 \in V_C$. Identify C as a subspace of V_C through the linear map $u \mapsto u(-1)\mathbf{1}$. There exists a unique vertex algebra structure on V_C with $\mathbf{1}$ as the vacuum vector and

with $Y(u,x) = u(x)$ for $u \in C$ (see [P]). Furthermore, C generates V_C as a vertex algebra. It was proved in [P] that for any linear map f from C into a vertex algebra V such that

$$fT(u) = \mathcal{D}f(u), \quad f(u_n v) = f(u)_n f(v) \quad \text{for } u, v \in C, \ n \geq 0,$$

f can be extended uniquely to a vertex-algebra homomorphism from V_C to V.

An *automorphism* of a conformal Lie algebra C is a linear automorphism θ of C such that $T\theta = \theta T$ and $\theta Y_-(u,x)v = Y_-(\theta(u),x)\theta(v)$ for $u, v \in C$. We have the following straightforward analogue of Lemma 4.1:

Lemma 4.13. *Let C be a conformal Lie algebra and let H be a group acting on C by automorphisms such that for any $a, b \in C$, $Y_-(hu,x)v = 0$ for all but finitely many $h \in H$. Then the linear map $Y_-^H : C \to \mathrm{Hom}(C, x^{-1}C[x^{-1}])$, defined by*

$$Y_-^H(u,x)v = \sum_{h \in H} Y_-(R_h u, x)v$$

for $u, v \in C$, naturally gives rise to a conformal Lie algebra structure on the quotient space, denoted by C/H, of C modulo the subspace linearly spanned by the vectors $R_h a - a$ for $h \in H$, $a \in C$.

The following notion, which is parallel to the notion of Γ-vertex algebra, extends the notion of Γ-conformal algebra in [GKK]:

Definition 4.14. Let Γ be a group as before. A Γ-*conformal Lie algebra* is a conformal Lie algebra (C, Y_-, T) equipped with group homomorphisms

$$R : \Gamma \to \mathrm{GL}(C); \ g \mapsto R_g,$$
$$\phi : \Gamma \to \mathbb{C}^\times$$

such that for any $a, b \in C$,

$$TR_g = \phi(g) R_g T, \tag{4.22}$$
$$R_g Y_-(a,x) R_{g^{-1}} = Y_-(R_g a, \phi(g)^{-1} x), \tag{4.23}$$
$$Y_-(R_g a, x)b = 0 \quad \text{for all but finitely many } g \in \Gamma. \tag{4.24}$$

Notice that in terms of components, (4.23) amounts to

$$R_g(u_m v) = \phi(g)^{m+1} (R_g u)_m R_g v \quad \text{for } m \in \mathbb{Z}.$$

We have the following analogue of Proposition 4.4:

Proposition 4.15. *Let Γ be a group and let C be a Γ-conformal Lie algebra. Define a bilinear multiplication on $C \otimes \mathbb{C}[t, t^{-1}]$ by*

$$[u \otimes t^m, v \otimes t^n]_\Gamma = \sum_{g \in \Gamma} \sum_{i \geq 0} \binom{m}{i} \phi(g)^{m+1} \left((R_g u)_i v \otimes t^{m+n-i} \right) \tag{4.25}$$

for $u, v \in C,\ m, n \in \mathbb{Z}$. Then the subspace linearly spanned by the elements

$$T(u) \otimes t^m + mu \otimes t^{m-1},\ \ \phi(g)^{m+1} R_g u \otimes t^m - u \otimes t^m$$

for $g \in \Gamma,\ u \in C,\ m \in \mathbb{Z}$ is a two-sided ideal of the nonassociative algebra $C \otimes \mathbb{C}[t, t^{-1}]$ and the quotient algebra is a Lie algebra, which we denote by $\hat{C}[\Gamma]$.

Proof. Associated to the conformal Lie algebra C, we have a Lie algebra $\mathcal{L}(C)$. Let Γ act on $C \otimes \mathbb{C}[t, t^{-1}]$ by

$$g(u \otimes t^m) = \phi(g)^{m+1}(R_g u \otimes t^m) \quad \text{for } g \in \Gamma,\ u \in C,\ m \in \mathbb{Z}.$$

For $g \in \Gamma,\ u, v \in C,\ m, n \in \mathbb{Z}$, we have

$$\begin{aligned}
[g(u \otimes t^m), g(v \otimes t^n)] &= \sum_{i \in \mathbb{N}} \binom{m}{i} \phi(g)^{m+n+2} (R_g u)_i (R_g v) \otimes t^{m+n-i} \\
&= \sum_{i \in \mathbb{N}} \binom{m}{i} \phi(g)^{m+n+1-i} R_g(u_i v) \otimes t^{m+n-i} \\
&= g[u \otimes t^m, v \otimes t^n].
\end{aligned}$$

Furthermore, using the relation (4.22) we have

$$\begin{aligned}
g(T \otimes 1 + 1 \otimes d/dt)(u \otimes t^m) &= g(Tu \otimes t^m + u \otimes mt^{m-1}) \\
&= \phi(g)^{m+1} R_g T u \otimes t^m + \phi(g)^m R_g u \otimes mt^{m-1} \\
&= \phi(g)^m T R_g u \otimes t^m + \phi(g)^m R_g u \otimes mt^{m-1} \\
&= \phi(g)^{-1}(T \otimes 1 + d/dt) g(u \otimes t^m).
\end{aligned}$$

It follows that Γ naturally acts on the Lie algebra $\mathcal{L}(C)$ by automorphisms. We have

$$\begin{aligned}
\sum_{g \in \Gamma} [g(u \otimes t^m), v \otimes t^n] &= \sum_{g \in \Gamma} \phi(g)^{m+1} [R_g u \otimes t^m, v \otimes t^n] \\
&= \sum_{g \in \Gamma} \sum_{i \in \mathbb{N}} \binom{m}{i} \phi(g)^{m+1} ((R_g u)_i v \otimes t^{m+n-i}) \\
&= [u \otimes t^m, v \otimes t^n]_\Gamma.
\end{aligned}$$

Now it follows immediately from Lemma 4.1 with $K = \mathcal{L}(C)$. $\square$

Lemma 4.16. *Let C be a Γ-conformal algebra and let V_C be the associated vertex algebra. Then the group action of Γ on C can be extended uniquely to a group action of Γ on V_C such that V_C becomes a Γ-vertex algebra.*

Proof. As C generates V_C as a vertex algebra, the uniqueness is clear. In the proof of Proposition 4.15 we have proved that for $g \in \Gamma$, the action of g on $C \otimes \mathbb{C}[t, t^{-1}]$, defined by

$$g(u \otimes t^m) = \phi(g)^{m+1}(R_g u \otimes t^m) \quad \text{for } u \in C,\ m \in \mathbb{Z},$$

reduces to an automorphism of the associated Lie algebra $\mathcal{L}(C)$. Clearly, g preserves the polar decomposition. Then g gives rise to a linear automorphism, denoted by R_g, of V_C with $g(\mathbf{1}) = \mathbf{1}$ and we have

$$R_g a_m v = \phi(g)^{m+1}(R_g a)_m R_g v \quad \text{for } a \in C,\ v \in V_C,\ m \in \mathbb{Z}.$$

As C generates V_C as a vertex algebra, it follows from induction (and the Jacobi identity of vertex algebra V_C) that

$$R_g u_m v = \phi(g)^{m+1}(R_g u)_m R_g v \quad \text{for all } u, v \in V_C.$$

It is easy to see that this defines a group action of Γ on V_C. Therefore, V_C is a Γ-vertex algebra. $\square$

Let C be a Γ-conformal Lie algebra. Set $H = \ker \phi \subset \Gamma$. Notice that for any $h \in H$, R_h is an automorphism of conformal Lie algebra C. Thus H acts on C by automorphisms. By Lemma 4.13, we have a conformal Lie algebra C/H. It is clear that C/H with the natural Γ/H-action is also a Γ/H-conformal Lie algebra. By Lemma 4.16, $V_{C/H}$ is naturally a Γ/H-vertex algebra. We define a notion of restricted module for the Lie algebra $\hat{C}[\Gamma]$ in the obvious way and for a restricted module W we define the notion $a_W(x)$ for $a \in C$ in the obvious way. With all these, by slightly modifying the proof of Theorem 4.9 we have:

Theorem 4.17. *Let Γ be a group and let C be a Γ-conformal Lie algebra. Set $H = \ker \phi \subset \Gamma$. Then any restricted module W for the Lie algebra $\hat{C}[\Gamma]$ is naturally a $V_{C/H}$-quasimodule with $Y_W(a, x) = a_W(x)$ for $a \in C$. On the other hand, any $V_{C/H}$-quasimodule W is naturally a restricted module for the Lie algebra $\hat{C}[\Gamma]$ with $a_W(x) = Y_W(a, x)$ for $a \in C$.*

References

[AABGP] B. N. Allison, S. Azam, S. Berman, Y. Gao and A. Pianzola, Extended affine Lie algebras and their root systems, *Memoirs Amer. Math. Soc.* **126**, 1997.

[BDM] K. Barron, C. Dong and G. Mason, Twisted sectors for tensor product vertex operator algebras associated to permutation groups, *Commun. Math. Phys.* **227** (2002), 349-384.

[BBS] S. Berman, Y. Billig and J. Szmigielski, Vertex operator algebras and the representation theory of toroidal algebras, in: *Recent Developments in Infinite-Dimensional Lie Algebras and Conformal Field Theory (Charlottesville, VA, 2000)*, Contemporary Math. **297**, Amer. Math. Soc., Providence, 2002, 1-26.

[BGT] S. Berman, Y. Gao and S. Tan, A unified view of some vertex operator constructions, *Israel J. Math.* **134** (2003), 29-60.

[DL] C. Dong and J. Lepowsky, *Generalized Vertex Algebras and Relative Vertex Operators*, Progress in Math., Vol. 112, 1993.

[DLM] C. Dong, H.-S. Li and G. Mason, Vertex Lie algebra, Poisson algebras and vertex operator algebras, in: *Recent Developments in Infinite-Dimensional Lie Algebras and Conformal Field Theory, Proc. of an International Conference, May 23-27, 2000, University of Virginia,* ed. by S. Berman, P. Fendley, Y.-Z. Huang, K. Misra and B. Parshall, Contemporary Math. **297**, Amer. Math. Soc., 2002, 69-96,

[FHL] I. B. Frenkel, Y.-Z. Huang and J. Lepowsky, On axiomatic approaches to vertex operator algebras and modules, Memoirs Amer. Math. Soc. **104**, 1993.

[FLM] I. B. Frenkel, J. Lepowsky and A. Meurman, *Vertex Operator Algebras and the Monster,* Pure and Applied Math., Vol. 134, Academic Press, Boston, 1988.

[FZ] I. Frenkel and Y.-C. Zhu, Vertex operator algebras associated to representations of affine and Virasoro algebras, *Duke Math. J.* **66** (1992), 123-168.

[GKK] M. Golenishcheva-Kutuzova and V. Kac, Γ-conformal algebras, *J. Math. Phys.* **39** (1998), 2290-2305.

[GKL1] M. Golenishcheva-Kutuzova and D. Lebedev, Vertex operator representation of some quantum tori Lie algebras, *Commun. Math. Phys.* **148** (1992), 403-416.

[GKL2] M. Golenishcheva-Kutuzova and D. Lebedev, $\mathbb{Z}$-graded trigonometric Lie subalgebras in $\widehat{A}_\infty$, $\widehat{B}_\infty$, $\widehat{C}_\infty$, and $\widehat{D}_\infty$, and their vertex operator representations, *J. Func. Anal. Application* **27** (1993), 12-24.

[K1] V. G. Kac, *Infinite-dimensional Lie Algebras*, 3rd ed., Cambridge Univ. Press, Cambridge, 1990.

[K2] V. G. Kac, *Vertex Algebras for Beginners*, University Lecture Series, Vol. 10, Amer. Math. Soc., 1997.

[LL] J. Lepowsky and H.-S. Li, *Introduction to Vertex Operator Algebras and Their Representations*, Progress in Math. Vol. 227, Birkhäuser, Boston, 2004.

[Li1] H.-S. Li, Local systems of vertex operators, vertex superalgebras and modules, *J. Pure Appl. Alg.* **109** (1996), 143-195.

[Li2] H.-S. Li, Local systems of twisted vertex operators, vertex superalgebras and twisted modules, in: *Moonshine, the Monster and related topics, Proc. Joint Summer Research Conference (Mount Holyoke, 1994)*, Contemporary Math. **193**, Amer. Math. Soc., Providence, 1996, 203-236.

[Li3] H.-S. Li, A new construction of vertex algebras and quasi modules for vertex algebras, *Adv. Math.* **202** (2006), 232-286.

[Li4] H.-S. Li, On twisted modules and quasimodules for vertex operator algebras, to appear in the Proceedings of the conference in North Carolina State University, Raleigh, 2005, in honor of Professors James Lepowsky and Robert Lee Wilson,

[Lia] B.-H. Lian, On the classification of simple vertex operator algebras, *Commun. Math. Phys.* **163** (1994), 307-357.

[MY] J. Morita and Y. Yoshii, Universal central extensions of Chevalley algebras over Laurent polynomial rings and GIM Lie algebras, *Proc. Japan Acad.* **61 Ser. A** (1985), 179-181.

[P] M. Primc, Vertex algebras generated by Lie algebras, *J. Pure Appl. Alg.* **135** (1999), 253-293.

[S1] K. Saito, Extended affine root systems 1 (Coxeter transformations), Publ. RIMS., Kyoto Univ. **21** (1985), 75-179.

[S2] K. Saito, Extended affine root systems 2 (flat invariants), Publ. RIMS., Kyoto Univ. **26** (1985), 15-78.